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Angle-variable holonomy in adiabatic excursion of  
an integrable Hamiltonian,  
JH Hannay, *J. Phys. A* **18**, 221–23 (1985)

It is an elementary fact that a one-freedom, time-independent Hamiltonian system for which the level sets of the Hamiltonian are bounded – a particle in a potential well, for example – undergoes periodic motion. However, if some parameters in the Hamiltonian are made to vary in time, the dynamics can become much more complicated. Much of the simplicity of the original periodic dynamics is restored, though, if the parameters change only a little over each period. In this case, the dynamics is nearly periodic; at any given time, the system nearly follows one of the orbits of the fixed-parameter Hamiltonian, specifically the orbit that encloses a phase space area of a fixed size determined by the initial conditions. This is the content of the adiabatic theorem in classical mechanics. Note that in general, energy is not conserved as the parameters change; work may be done on or by the system. The adiabatically conserved quantity is precisely the phase space area enclosed by the orbit, or equivalently, the action. The argument extends to more degrees of freedom provided the Hamiltonian is integrable, although the adiabatic dynamics may be more complicated in higher dimensions, and the adiabatic invariants not as nearly conserved.

John Hannay’s 1985 paper [4] provided a refinement of the classical adiabatic theorem. Suppose that over a long time  $T$ , the Hamiltonian is taken around a given cycle  $C$  in parameter space, so that the initial and final Hamiltonians are the same. According to the adiabatic theorem, the system (nearly) returns to the orbit it started upon. Hannay posed the question, to which point along this orbit does it return? Or, in terms of action-angle variables, by how much has the angle changed over the course of the cycle? The obvious, and largest, contribution, the so-called dynamical angle, is the time integral of the (slowly varying) frequency, and is of order  $T$ . Hannay found an additional contribution, now called the Hannay angle, which depends only on the cycle  $C$  and not on the rate at which it is traversed. It is,

therefore, a purely geometrical quantity. According to one formula [2], the Hannay angle is given by the integral of a two-form  $V$  through a surface  $S$  in parameter space bounded by  $C$ ,

$$V(I, R) = \frac{\partial}{\partial I} \langle dq \wedge dp \rangle. \quad (1)$$

Here,  $q(\theta, I, R)$ ,  $p(\theta, I, R)$  denotes the canonical transformation to action-angle coordinates, and the averaging  $\langle \cdot \rangle$  is with respect to  $\theta$ .

The precession of a (spherical) Foucault pendulum provides a familiar example of the Hannay angle. The direction of the Earth's gravitational field at the pendulum, referred to an inertial frame fixed in space, slowly executes a cycle as the Earth rotates around its axis over the course of a day. If the pendulum starts off swinging in a plane, the associated dynamical angle turns out to be zero, and the daily precession of the plane of oscillation is given just by the Hannay angle, which turns out to be  $2\pi$  minus the solid angle swept out by the pendulum axis, or  $2\pi \sin \alpha$ , where  $\alpha$  is the Earth's latitude at the pendulum. (This solid angle formula also holds for more general planetary rotations than the Earth's, in which the axis and rate of rotation vary in time.)

Hannay's discovery was motivated by Michael Berry's discovery the previous year of the geometric, or Berry, phase in quantum mechanics [1]. The Berry phase is a phase  $\gamma_n$  accumulated by the eigenstates  $|n\rangle$  of slowly cycled quantum Hamiltonians, which appears as a correction to the dynamical phase given by the integral of the (slowly varying) energy  $E_n$  divided by  $\hbar$ . Hannay, Berry's colleague at the University of Bristol, wondered whether there might be a classical analogue of the Berry phase, and was led to the angle-variable holonomy. The connection between the Hannay angle and the Berry phase turns out to be more than just an analogy. In the semiclassical limit, at least for quantum Hamiltonians which are classically integrable, the Hannay angle stands in the same relation to the Berry phase as does the classical frequency and the Bohr frequency [2], namely

$$\omega(I) = \frac{E_{n+1} - E_n}{\hbar}, \quad \text{and} \quad \theta_H(I) = \gamma_{n+1} - \gamma_n, \quad (2)$$

where the quantum number  $n$  is related to the classical action  $I$  by the semiclassical quantization condition.

The Hannay angle and Berry phase turn out to be ubiquitous across physics, appearing in polarisation optics, molecular spectroscopy, condensed matter, cold atoms, plasma physics, fluid dynamics and celestial mechanics. One reason is that so many physical phenomena involve the coupling of systems (or distinct degrees of freedom) evolving on different time-scales. In molecular physics, for example, the

electrons comprise the fast system, and the nuclei, the slow system – this is the framework for the Born-Oppenheimer approximation. The Hannay angle and Berry phase describe effects induced by the slow system on the fast system. If the slow system is allowed to evolve according to its own dynamics (rather than having its evolution prescribed in advance), there is a reciprocal effect of the fast system on the slow, called geometric magnetism [3], which provides a correction to the leading-order (eg Born-Oppenheimer) slow Hamiltonian. For simplicity, if we take the slow variables  $\mathbf{R}$  to be three dimensional, then the Hannay two-form (1) can be regarded as a vector field parameterised by the (nearly constant) action of the fast system, which acts as an effective magnetic field in the slow dynamics. Like magnetism itself, geometric magnetism requires a breaking of time-reversal symmetry. Thus, the Hannay angle and Berry phase reveal a mechanism by which gauge fields can arise in classical and quantum systems.

There is another class of problems where the Hannay angle has proved to be a unifying concept. These involve phenomena where symmetries permit a reduction to a lower-dimensional description. Hannay angles appear in the reconstruction of the full dynamics from the reduced dynamics [5]. Examples include semirigid bodies, whereby a cycle of internal deformations can produce a change in orientation without any angular momentum (falling cats landing upright), and swimming at low Reynolds number, whereby a cycle of internal deformations (a swimming stroke) can produce a change in position (locomotion) without any linear momentum [6].

## References

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